Abundant New Multiple Soliton-like Solutions and Rational Solutions of the (2+1)-Dimensional Broer-Kaup Equation

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In this paper we firstly improve the homogeneous balance method due to Wang, which was only used to obtain single soliton solutions of nonlinear evolution equations, and apply it to (2+1)-dimensional Broer-Kaup (BK) equations such that a Backlund transformation is found again. Considering further the obtained Backlund transformation, the relations are deduced among BK eqautions, well-known Burgers equations and linear heat equations. Finally, abundant multiple soliton-like solutions and infinite rational solutions are obtained from the relations.

Key words: (2+1)-dimensional Broer-Kaup equation; Backlund Transformation; Burgers Equation; Soliton Solution; Rational solution.

1. Introduction

Presently more attention is paid on the study of Backlund transformations, Lax pairs, symmetries, conservation laws, Painleve tests, bi-Hamiltonian structures, integrability, similarity reduction, exact solutions, etc. for nonlinear evolution equations in soliton theory and dynamical systems [1]. Recently, starting from the symmetry of the KP equation [2, 3], the (2+1)-dimensional Broer-Kaup equation [2, 3]

$$H_{ty} + 2(HH_x)_y + 2G_{xx} - H_{xxy} = 0, \qquad (1)$$

$$G_t + 2(GH)_x + G_{xx} = 0 \qquad (2)$$

was presented. The Painleve property and infinity of many truncated symmetries with arbitrary functions of t are, respectively, discussed [4] for (1) and (2) by using the WTC approach [5] and the formal series symmetry method [6]. Recently we obtained some single soliton-like solutions of (1) and (2) [7].Wang presented a powerful homogeneous balance method to obtain solitary wave solutions of the given NEEs. However, Wang used this method only to find single solitary wave solutions [8]. Zhang [9] extended the homogeneous balance method to dispersive long-wave equations so that multiple soliton solutions are found. We extended the method to the KPP equation, WBK equation, etc., so that some new results were derived [10, 11]. A natural problem is whether there are other types of solutions based on the homogeneous balance method. To solve the problem, we must seek new transformations by which more new exact solutions can be obtained.

In this paper, we would like to further generalize this method [8] and apply it to investigate the Backlund transformation, non-local symmetry and multiple soliton-like solutions of the (2+1)-dimensional Broer-Kaup equation (1) and (2). The method is remarkably straightforward and can also be applied to other equations.

2. Leading to a Backlund Transformation and Some Relations

By using the idea of the WTC method [5] or the homogeneous balance method [8 - 11], in order to balance the terms H_{xxy}, $(HH_x)_y$ and G_{xx} in (1) and the term G_{xx} and $(GH)_x$ in (2), respectively, we suppose that (1) and (2) have solutions in the forms

$$H(x,y,t) = \frac{\partial}{\partial x} h[w(x,y,t)] + H_0 = h' w_x + H_0, \qquad (3)$$

$$G(x,y,t) = \frac{\partial^2}{\partial x \partial y} g[w(x,y,t)] + G_0$$
$$= g'' w_x w_y + g' w_{xy} + G_0, \qquad (4)$$

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in which the functions $H(x, y, t)$ and $G(x, y, t)$ are expressed by two functions $h(w)$ and $g(w)$ of one argument $w(x, y, t)$ only, where $h(w)$, $g(w)$, $H_0 = H_0(x, y, t)$ and $G_0 = G_0(x, y, t)$ are functions to be determined later. Next we would like to know if the nonlinear and dispersive effects in (1) and (2) can be partially balanced by using (3) and (4). Thus the task in the following is to seek for the functions $h(w)$, $g(w)$, $w(x, y, t)$, H_0 and G_0 such that (3) and (4) actually satisfy (1) and (2).

With the aid of Mathematica, we can easily deduce that from (3) and (4)

$$H_{ty} = h'''w_xw_yw_t + h''w_{xy}w_t + h''w_xw_{ty} + h''w_yw_{xt} + h'w_{xyt} + H_{0ty}, \tag{5}$$

$$\begin{aligned}(HH_x)_y = H_xH_y + HH_{xy} = (h''^2 + h'h''')w_x^3w_y + 3h'h''w_x^2w_{xy} + 2h'h''w_xw_{xx}w_y + h''H_{0y}w_x^2 \\ + h'^2w_{xx}w_{xy} + h'^2w_xw_{xxy} + h'''H_0w_x^2w_y + 2h''H_0w_xw_{xy} + h''H_0w_{xx}w_y \\ + h'H_0w_{xxy} + h'H_{0y}w_{xx} + h''H_{0x}w_xw_y + h'H_{0x}w_{xy} + h'H_{0xy}w_x + H_{0x}H_{0y},\end{aligned} \tag{6}$$

$$G_{xx} = g''''w_x^3w_y + 3g'''w_x^2w_{xy} + 3g'''w_xw_yw_{xx} + 3g''w_{xy}w_{xx} + 3g''w_xw_{xxy} + g''w_yw_{xxx} + g'w_{xxxy} + G_{0xx}, \tag{7}$$

$$H_{xxy} = h''''w_x^3w_y + 3h'''w_x^2w_{xy} + 3h'''w_xw_yw_{xx} + 3h''w_{xy}w_{xx} + 3h''w_xw_{xxy} + h''w_yw_{xxx} + h'w_{xxxy} + H_{0xxy}, \tag{8}$$

$$G_t = g'''w_xw_yw_t + g''w_{xy}w_t + g''w_xw_{ty} + g''w_yw_{xt} + g'w_{xyt} + G_{0t}, \tag{9}$$

$$\begin{aligned}(GH)_x = G_xH + GH_x = (h'g''' + g''h'')w_x^3w_y + (2h'g'' + h''g')w_x^2w_{xy} + 2h'g''w_xw_yw_{xx} + h'g'w_xw_{xxy} \\ + g'''H_0w_x^2w_y + 2g''H_0w_xw_{xy} + g''H_0w_{xx}w_y + g'H_0w_{xxy} + h'g'w_{xx}w_{xy} + h'G_0w_x^2 + h'G_0w_{xx} \\ + h'G_{0x}w_x + g''H_{0x}w_xw_y + g'H_{0x}w_{xy} + G_0H_{0x} + H_0G_{0x}.\end{aligned} \tag{10}$$

Substituting (5 - 10) into (1) and (2), and collecting all the homogeneous terms in partial derivatives of $w(x, y, t)$, we get

$$\begin{aligned}&[-h'''' + 2(h''^2 + h'h''') + 2g'''']w_x^3w_y + [-3h'''w_x^2w_{xy} - 3h'''w_xw_yw_{xx} + 2(3h'h''w_x^2w_{xy} + 2h'h''w_xw_yw_{xx} \\ &+ h'''H_0w_x^2w_y) + 6g'''w_x^2w_{xy} + 6g'''w_xw_yw_{xx} + h'''w_xw_yw_t] + [-3h''w_{xx}w_{xy} - 3h''w_xw_{xxy} - h''w_yw_{xxx} \\ &+ 2(h''H_{0y}w_x^2 + h'^2w_{xx}w_{xy} + h''H_{0x}w_xw_y + h'^2w_xw_{xxy} + 2h''H_0w_xw_{xy} + h''H_0w_yw_{xx}) + 6g''w_{xy}w_{xx} \\ &+ 6g''w_xw_{xxy} + 2g''w_yw_{xxx} + h''w_{xy}w_t + h''w_xw_{yt} + h''w_yw_{xt}] + [-h'w_{xxxy} + 2h'(H_{0y}w_{xx} + H_{0x}w_{xy} \\ &+ H_{0xy}w_x + H_0w_{xxy} + 2g'w_{xxxy} + h'w_{xyt}] + H_{0ty} + 2G_{0xx} + 2H_{0x}H_{0y} + 2H_0H_{0xy} - H_{0xxy} = 0.\end{aligned} \tag{11}$$

$$\begin{aligned}&[g'''' + 2(h'g''' + h''g'')]w_x^3w_y + [3g'''w_x^2w_{xy} + 3g'''w_xw_yw_{xx} + 2(2h'g''w_x^2w_{xy} + 2h'g''w_xw_yw_{xx} \\ &+ g'''H_0w_x^2w_y + h''g'w_x^2w_{xy}) + g'''w_xw_yw_t] + [3g''w_{xx}w_{xy} + 3g''w_xw_{xxy} + g''w_yw_{xxx} + 2(h'g'w_{xx}w_{xy} \\ &+ h'^2w_xw_{xxy} + 2g''H_0w_xw_{xy} + g''H_0w_yw_{xx} + h''G_0w_x^2 + g''H_{0x}w_xw_y) + g''w_{xy}w_t + g''w_xw_{yt} + g''w_yw_{xt}] \\ &+ [g'w_{xxxy} + 2(h'G_{0x}w_x + g'H_{0x}w_{xy} + g'H_0w_{xxy} + g'w_{xyt} + h'G_0w_{xx})] \\ &+ G_{0t} + 2G_{0xx} + 2G_{0x}H_0 + 2G_0H_{0x} = 0.\end{aligned} \tag{12}$$

From (11) and (12) we can learn that the nonlinear terms and the highest order partial derivatives terms in (1) and (2) have been balanced by using (3) and (4).

To determine the functions $h(w)$ and $g(w)$, we set the coefficients of the term $w_x^3 w_y$ in (11) and (12) to zero, respectively, yields a system of order differential equations of $h(w)$ and $g(w)$

$$\begin{cases} -h'''' + 2(h''^2 + h'h''') + 2g'''' = 0, \\ g'''' + 2(h'g''' + h''g'') = 0. \end{cases} \tag{13}$$

which have the solutions

$$\begin{cases} h(w) = \ln w(x,y,t), \\ g(w) = \ln w(x,y,t), \end{cases} \tag{14}$$

for which, this we have the following relations

$$h'h'' = -\frac{1}{2}h''', \; g''' = h''', \; h'^2 = -h'', \; g' = h', \tag{15}$$

$$g'' = h'', h'g'' = -\frac{1}{2}g''', h''g' = -\frac{1}{2}g''', h'g' = -g''.$$

According to the relations (13 - 15), (11) and (12) reduce respectively to

$$[w_x w_y(w_{xx} + 2H_0 w_x + w_t)]h''' + [w_{xy}(w_{xx} + 2H_0 w_x + w_t) + w_x\frac{\partial}{\partial y}(w_{xx} + 2H_0 w_x + w_t) + w_y\frac{\partial}{\partial x}(w_{xx}$$

$$+ 2H_0 w_x + w_t)]h'' + (w_{xx} + 2H_0 w_x + w_t)_{xy}h' + (H_{0ty} + 2G_{0xx} + 2H_{0x}H_{0y} + 2H_0 H_{0xy} - H_{0xxy})h^0 = 0 \tag{16}$$

and

$$[w_x w_y(w_{xx} + 2H_0 w_x + w_t)]g''' + [w_{xy}(w_{xx} + 2H_0 w_x + w_t) + w_x(w_{xxy} + 2H_0 w_{xy} + w_{ty} + 2G_0 w_x)$$

$$+ w_y\frac{\partial}{\partial x}(w_{xx} + 2H_0 w_x + w_t)]g'' + [w_{xxxy} + w_{xyt} + 2(H_0 w_{xxy} + G_{0x}w_x + G_0 w_{xx} + H_{0x}w_{xy}]g'$$

$$+ (G_{0t} + 2G_{0xx} + 2G_{0x}H_0 + 2G_0 H_{0x})g^0 = 0. \tag{17}$$

Because the two sets of $h''' = \frac{2}{w^3}$, $h'' = -\frac{1}{w^2}$, $h' = \frac{1}{w}$, $h^0 = 1$ and $g''' = \frac{2}{w^3}$, $g'' = -\frac{1}{w^2}$, $g' = \frac{1}{w}$, $g^0 = 1$ are linearly independent, respectively, we set the coefficients of h''', h'', h' and h^0 in (16) and the coefficients of g''', g'', g' and g^0 in (17) to zero, respectively, such that the following system of equations with respect to $w(x,y,t)$ is given:

$$w_x w_y(w_{xx} + 2H_0 w_x + w_t) = 0, \tag{18}$$

$$w_{xy}(w_{xx} + 2H_0 w_x + w_t) + w_x(w_{xx} + 2H_0 w_x + w_t)_y + w_y(w_{xx} + 2H_0 w_x + w_t) = 0, \tag{19}$$

$$(w_{xx} + 2H_0 w_x + w_t)_{xy} = 0, \tag{20}$$

$$H_{0ty} + 2G_{0xx} + 2H_{0x}H_{0y} + 2H_0 H_{0xy} - H_{0xxy} = 0, \tag{21}$$

$$w_{xy}(w_{xx} + 2H_0 w_x + w_t) + w_x(w_{xxy} + 2H_0 w_{xy} + 2G_0 w_x + w_{ty}) + w_y(w_{xx} + 2H_0 w_x + w_t) = 0, \tag{22}$$

$$w_{xxxy} + w_{xyt} + 2(H_0 w_{xxy} + H_{0x}w_{xy} + G_{0x}w_x + G_0 w_{xx} = 0, \tag{23}$$

$$G_{0t} + 2G_{0xx} + 2G_{0x}H_0 + 2G_0 H_{0x} = 0. \tag{24}$$

From (18 - 24), we can reduce them as follows:

$$w_x w_y(w_{xx} + 2H_0 w_x + w_t) = 0, \tag{25}$$

$$w_{xy}(w_{xx} + 2H_0 w_x + w_t) + w_x(w_{xx} + 2H_0 w_x + w_t)_y + w_y(w_{xx} + 2H_0 w_x + w_t) = 0, \tag{26}$$

$$(w_{xx} + 2H_0 w_x + w_t)_{xy} = 0, \tag{27}$$

$$G_0 = H_{0y}, \tag{28}$$

and $(H_0, G_0 = H_{0y})$ is a solution of (1) and (2). Therefore, substituting (14) into (3) and (4) yields the Backlund transformation

$$\begin{cases} H(x,y,t) = \dfrac{w_x}{w} + H_0(x,y,t), \\ G(x,y,t) = \dfrac{w_{xy}}{w} - \dfrac{w_x w_y}{w^2} + G_0(x,y,t), \end{cases} \tag{29}$$

where $w = w(x,y,t)$ satisfies (25 - 27).

In order to use the obtained transformation to seek exact solution of (1) and (2), we give two special cases.

Case 1: Setting $H_0 = h_0 =$ const., and $G_0 = 0$, from (25 - 29), we can derive the Backlund transformation

$$\begin{cases} H(x,y,t) = \dfrac{w_x}{w} + h_0, \\ G(x,y,t) = \dfrac{w_{xy}}{w} - \dfrac{w_x w_y}{w^2}, \end{cases} \tag{30}$$

which is a relation between (1), (2) and the linear generalized heat equation

$$w_t + 2h_0 w_x + w_{xx} = 0. \tag{31}$$

Case 2: Setting $H_0 = w$ and $G_0 = w_y$, from (25 - 29), we can derive another Backlund transformation

$$\begin{cases} H(x,y,t) = \dfrac{w_x}{w} + w, \\ G(x,y,t) = \dfrac{w_{xy}}{w} - \dfrac{w_x w_y}{w^2} + w_y, \end{cases} \tag{33}$$

which is a relation between (1), (2) and the well-known Burgers equation

$$w_t + 2ww_x + w_{xx} = 0. \tag{34}$$

Remark: If we know a solution of (31) or (33), then the corresponding solution of (1) and (2) can be obtained by using the Backlund transformation (30) or (32). In what follows, we would like to consider solutions of system (1) and (2) by using this idea.

3. Exact Solutions-Soliton Solution and Rational Solutions

Case A: From (31), we can deduce the special solution

$$w(x,y,t) = a_0(y) + \sum_{i=1}^{n} a_i(y)\exp[k_i(y)x + l_i(y) - (k_i^2(y) + 2h_0 k_i(y))t + c], \ n = 1,2,3,\dots, \tag{34}$$

which satisfies (25 - 27), where $k_i(y)$, $l_i(y)$, $a_0(y)$ and $a_i(y)$ ($i = 0,1,2,\dots n$) are arbitrary functions of y and c are all arbitrary constants. According the Backlund transformation (30), we obtain the new multiple soliton-like solutions of (1) and (2)

$$H = \frac{\sum_{i=1}^{n} a_i(y)k_i(y)\exp[k_i(y)x + l_i(y) - (k_i^2(y) + 2h_0 k_i(y))t + c]}{a_0(y) + \sum_{i=1}^{n} a_i(y)\exp[k_i(y)x + l_i(y) - (k_i^2(y) + 2h_0 k_i(y))t + c]} + h_0, \tag{35}$$

$$\begin{aligned} G = {} & \frac{\sum_{i=1}^{n}[a_i(y)k_i'(y)a_i'(y)k_i(y) + a_i(y)k_i(y)\xi_i'(y)]\exp(\xi_i)}{a_0(y) + \sum_{i=1}^{n} a_i(y)\exp[k_i(y)x + l(y) - (k_i^2(y) + 2H_0 k_i(y))t + c]} \\ & - \frac{\sum_{i=1}^{n} a_i(y)k_i(y)\exp(\xi_i)[a_0'(y) + \sum_{i=1}^{n}(a_i'(y) + a_i\xi_i'(y))\exp(\xi_i)]}{\{a_0(y) + \sum_{i=1}^{n} a_i \exp[k_i(y)x + l_i(y) - (k_i^2(y) + 2h_0 k_i(y))t + c]\}^2}, \end{aligned} \tag{36}$$

where

$$\xi_i = k_i(y)x + l_i(y) - (k_i(y)^2 + 2h_0 k_i(y))t + c, \quad \xi_i'(y) = k_i'(y)x - 2(k_i(y)k_i'(y) + h_0 k_i(y))t + l_i'(y).$$

If we take $k_i(y) = k_i =$ const. and $l_i(y) = l_i y$, $l_i =$ const. $a_j(y) =$ const. ($j = 0,1,2,\dots n$) in (35) and (36), we can find the new multiple soliton solutions of (1) and (2) as follows

$$H(x,y,t) = \frac{\sum_{i=1}^{n} a_i k_i \exp[k_i x + l_i y - (k_i^2 + 2h_0 k_i)t + c]}{1 + \sum_{i=1}^{n} a_i \exp[k_i x + l_i y - (k_i^2 + 2h_0 k_i)t + c]} + h_0,$$

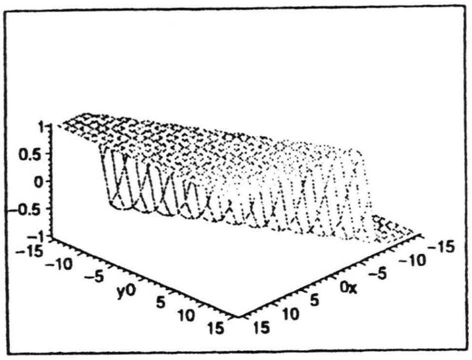

Fig. 1

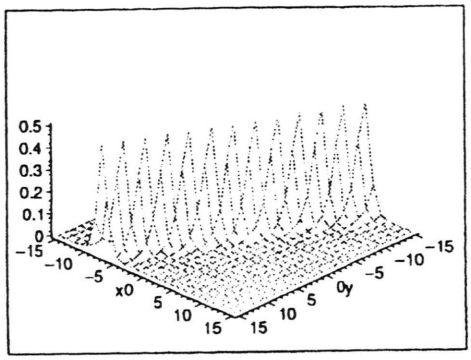

Fig. 2

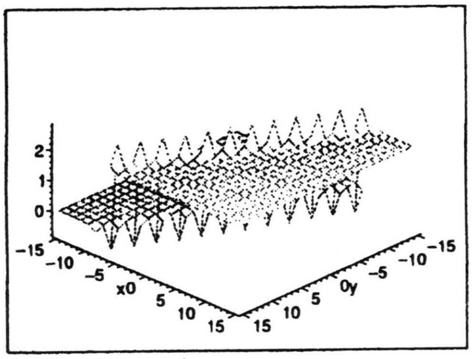

Fig. 3

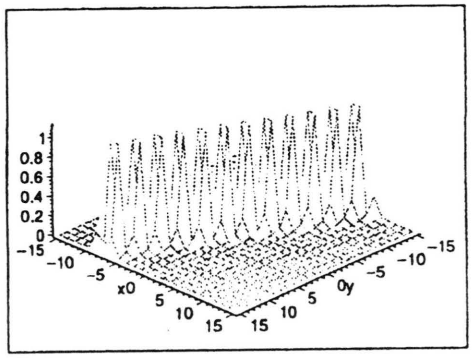

Fig. 4

$$G(x,y,t) = \frac{\sum_{i=1}^{n} a_i k_i l_i \exp[k_i x + l_i y - (k_i^2 + 2h_0 k_i)t + c]}{1 + \sum_{i=1}^{n} a_i \exp[k_i x + l_i y - (k_i^2 + 2h_0 k_i)t + c]}$$

$$- \frac{\sum_{i=1}^{n} a_i k_i \exp[k_i x + l_i y - (k_i^2 + 2h_0 k_i)t + c] \sum_{i=1}^{n} l_i a_i \exp[k_i x + l_i y - (k_i^2 + 2h_0 k_i)t + c]}{\{1 + \sum_{i=1}^{n} a_i \exp[k_i x + l_i y - (k_i^2 + 2h_0 k_i)t + c]\}^2}.$$

Case 1a: When $n = 1, a_1 > 0$, we obtain kink-shaped soliton solutions and bell-shaped soliton solutions of (1) an (2):

$$H_1(x,y,t) = \frac{k_1}{2}\tanh\{\frac{1}{2}[k_1 x + l_1 y - (k_1^2 + 2h_0 k_1)t + \ln a_1 + c]\} + \frac{k_1}{2} + h_0, \tag{37a}$$

$$G_1(x,y,t) = \frac{k_1 l_1}{4}\mathrm{sech}^2\{\frac{1}{2}[k_1 x + l_1 y - (k_1^2 + 2h_0 k_1)t + \ln a_1 + c]\}. \tag{37b}$$

Figure 1 is a plot of (37a) and shows a kink-shaped soliton solution in (2+1)-dimensions at $t = 0$ with $k_1 = 2$, $l_1 = 1$, $a_1 = 1$, $h_0 = c = 0$. Figure 2 is a plot of (37b) and shows a bell-shaped soliton solution in (2+1)-dimensions at $t = 0$ with the same conditions as Figure 1.

Case 1b: When $n = 1, a_1 < 0$, we can obtain singular soliton solutions for (1) and (2)

$$H_2(x,y,t) = \frac{k_1}{2}\coth\{\frac{1}{2}[k_1 x + l_1 y - (k_1^2 + 2h_0 k_1)t + \ln(-a_1) + c]\} + \frac{k_1}{2} + h_0, \tag{38a}$$

$$G_2(x,y,t) = \frac{k_1 l_1}{4} csch^2\{\frac{1}{2}[k_1 x + l_1 y - (k_1^2 + 2h_0 k_1)t + \ln(-a_1) + c]\}. \tag{38b}$$

Figure 3 is a plot of (38a) and shows a singular soliton solution in (2+1)-dimensions at $t = 0$ with $k_1 = 2$,

$l_1 = 1, a_1 = 1, h_0 = c = 0$. Figure 4 is a plot of (38b) and shows a singular soliton solution in (2+1)-dimensions at $t = 0$ with the same conditions as Figure 3. These singular solutions develop a singularity at a finite point, i. e., for fixed $t = 0$, there always exist $x = x_0$, $y = y_0$ at which the solutions blow up.

Case 2: As $n = 2$, we get double-soliton solutions of (1) an (2):

$$H_3(x,y,t) = \frac{a_1k_1\exp(\xi_1) + a_2k_2\exp(\xi_2)}{1 + a_1\exp(\xi_1) + a_2\exp(\xi_2)} + h_0, \tag{39a}$$

$$G_3(x,y,t) = \frac{a_1k_1l_1\exp(\xi_1) + a_2k_2l_2\exp(\xi_2)}{[1 + a_1\exp(\xi_1) + a_2\exp(\xi_2)]^2} + \frac{a_1a_2(k_1-k_2)(l_1-l_2)\exp(\xi_1-\xi_2)}{[1 + a_1\exp(\xi_1) + a_2\exp(\xi_2)]^2}. \tag{39b}$$

where $\xi_1 = k_1x + l_1y - (k_1^2 + 2h_0k_1)t + c$, $\xi_2 = k_2x + l_2y - (k_2^2 + 2h_0k_2)t + c$.

Figure 5 is a plot of (39a) and shows a soliton solution in (2+1)-dimensions at $t = 0$ with $k_1 = 1$, $k_2 = 2$, $l_1 = l_2 = -90$, $a_1 = a_2 = 1$, $h_0 = c = 0$. Figure 6 is a plot of (39b) and shows a singular soliton solution in (2+1)-dimensions at $t = 0$ with $k_1 = 1$, $k_2 = 2$, $l_1 = 2$, $l_2 = 1$, $a_1 = a_2 = 1$, $h_0 = c = 0$.

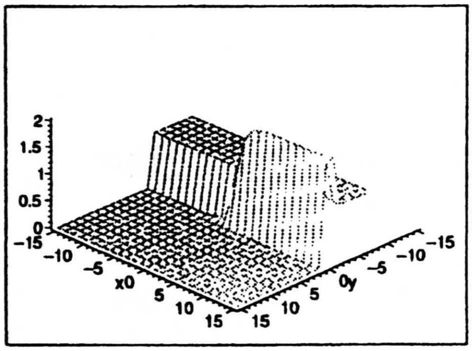

Fig. 5

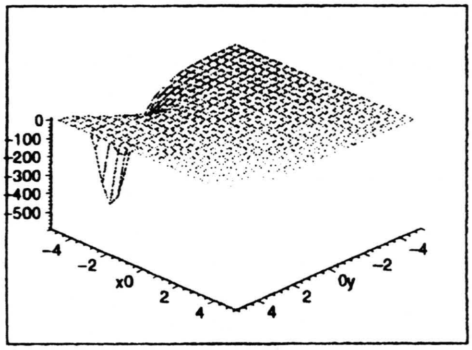

Fig. 6

Case 3: When $n = 3$, we get tri-soliton solutions of (1) an (2):

$$H_4(x,y,t) = \frac{a_1k_1\exp(\xi_1) + a_2k_2\exp(\xi_2) + a_3k_3\exp(\xi_3)}{1 + a_1\exp(\xi_1) + a_2\exp(\xi_2) + a_3\exp(\xi_3)} + h_0, \tag{40a}$$

$$\begin{aligned} G_4(x,y,t) = {} & \frac{a_1k_1l_1\exp(\xi_1) + a_2k_2l_2\exp(\xi_2) + a_3k_3\exp(\xi_3)}{[1 + a_1\exp(\xi_1) + a_2\exp(\xi_2) + a_3\exp(\xi_3)]^2} + \frac{a_1a_2(k_1-k_2)(l_1-l_2)\exp(\xi_1-\xi_2)}{[1 + a_1\exp(\xi_1) + a_2\exp(\xi_2) + a_3\exp(\xi_3)]^2} \\ & + \frac{a_1a_3(k_1-k_3)(l_1-l_3)\exp(\xi_1-\xi_3)}{[1 + a_1\exp(\xi_1) + a_2\exp(\xi_2) + a_3\exp(\xi_3)]^2} + \frac{a_2a_3(k_2-k_3)(l_2-l_3)\exp(\xi_2-\xi_3)}{[1 + a_1\exp(\xi_1) + a_2\exp(\xi_2) + a_3\exp(\xi_3)]^2}, \end{aligned}$$

where $\xi_3 = k_3x + l_3y - (k_3^2 + 2h_0k_3)t + c$.

The properties of the two solutions are similar to the solutions (39a) and (39b). For the cases $n > 3$, we omit them here.

Case B: We can easily get multiple soliton-like solutions of Burgers equations (33) in (2+1)-dimensional space:

$$w(x,y,t) = \frac{\sum_{i=1}^{n} a_i(y)k_i(y)\exp[k_i(y)x + l_i(y) - k_i^2(y)t]}{a_0(y) + \sum_{i=1}^{n} a_i(y)\exp[k_i(y)x + l_i(y) - k_i^2(y)t]}, \tag{41}$$

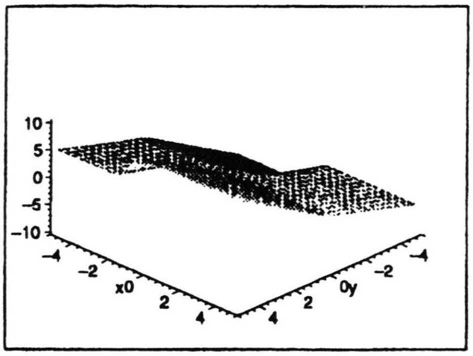

Fig. 7

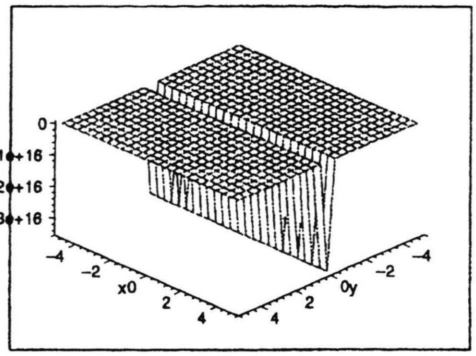

Fig. 8

where $k_i(y), l_i(y), a_i(y) (i = 0, 1, 2, ...n)$ are arbitrary functions of y. According the Backlund transformation (32), we obtain the new multiple soliton-like solutions of (1) and (2)

$$H = \frac{\sum_{i=1}^{n} a_i(y)k_i^2(y)\exp[k_i(y)x + l_i(y) - (k_i^2(y) + 2h_0k_i(y))t + c]}{\sum_{i=1}^{n} a_i(y)k_i\exp[k_i(y)x + l_i(y) - (k_i^2(y) + 2h_0k_i(y))t + c]} + h_0, \tag{42a}$$

$$G = \frac{\sum_{i=1}^{n}[2a_i(y)k_i'(y)k_i(y) + a_i'(y)k_i^2(y) + a_i(y)k_i^2(y)\xi_i'(y)]\exp(\xi_i)}{\sum_{i=1}^{n} a_i(y)k_i\exp[k_i(y)x + l_i(y) - (k_i^2(y) + 2h_0k_i(y))t + c]}$$
$$- \frac{[\sum_{i=1}^{n} a_i(y)k_i^2(y)\exp(\xi_i)][\sum_{i=1}^{n}[a_i(y)k_i'(y) + a_i'(y)k_i(y) + a_i(y)k_i(y)\xi_i'(y)]\exp(\xi_i)]}{[\sum_{i=1}^{n} a_i(y)k_i(y)\exp(\xi_i(y)]^2}. \tag{42b}$$

where $\xi_i = k_i(y)x + l_i(y) - (k_i(y)^2 + 2h_0k_i(y))t + c,\quad \xi_i' = k_i'(y)x - 2(k_i(y)k_i'(y) + h_0k_i(y))t + l_i'(y),$

In particular, when $n = 2$, the solutions (42a) and (42b) become

$$H_5 = \frac{a_1(y)k_1^2(y)\exp(\xi_1) + a_2(y)k_2^2(y)\exp(\xi_2)}{a_1(y)k_1\exp(\xi_1) + a_2(y)k_2\exp(\xi_2)} + h_0, \tag{43a}$$

$$G = \frac{A_1(x,y)\exp(\xi_1) + A_2(x,y)\exp(\xi_2)}{a_1(y)k_1\exp(\xi_1) + a_2(y)k_1(y)\exp(xi_2)}$$
$$- \frac{[a_1(y)k_1^2(y)\exp(\xi_1) + a_2(y)k_2^2(y)\exp(\xi_2)][B_1(x,y)\exp(\xi_1) + B_2(x,y)\exp(\xi_2)]}{[a_1(y)k_1(y)\exp(\xi_1) + a_2(y)k_2(y)\exp(\xi_2)]^2}, \tag{43b}$$

where

$$A_i(x,y) = 2a_i(y)k_i'(y)k_i(y) + a_i'(y)k_i^2(y) + a_i(y)k_i^2(y)\xi_i',\quad B_i(x,y) = a_i(y)k_i'(y) + a_i'(y)k_i(y) + a_i(y)k_i(y)\xi_i',$$

$$\xi_i = k_i(y)x + l_i(y) - (k_i(y)^2 + 2h_0k_i(y))t + c,\quad \xi_i' = k_i'(y)x - 2(k_i(y)k_i'(y) + h_0k_i(y))t + l_i'(y),\quad (i = 1, 2). \tag{43c}$$

Figure 7 is a plot of (43a) and shows a solution in (2+1)-dimensions at $t = 0$ with $k_1(y) = 2y$, $k_2(y) = y$, $l_1(y) = y$, $l_2(y) = \frac{1}{10}y$, $a_1 = a_2 = 1$, $h_0 = c = 0$. Figure 8 is a plot of (43b) and shows a singular soliton solution in (2+1)-dimensions at $t = 0$ with $k_1(y) = 2y$, $k_2(y) = y$, $l_1(y) = 2y$,$l_2(y) = y$, $a_1 = a_2 = 1$, $h_0 = c = 0$.

Case C: When $H_0 = h_0 = 0$, we set (31) has the solution:

$$\phi(x,y,t) = \sum_{i=0}^{n} f_i(x,y)t^i = f_n(x,y)t^n + \cdots + f_1(x,y)t + f_0(x,y), \tag{44}$$

where $f_{nxx}(x,y) = 0$ and

$$nf_n(x,y) + f_{n-1,xx}(x,y) = 0,\ (n-1)f_{n-1}(x,y) + f_{n-2,xx}(x,y) = 0,\ \cdots,\ f_1(x,y) + f_{0xx}(x,y) = 0, \tag{45}$$

which gives rise to

$$f_i(x,y) = (-1)^{n-i}(n-i)!\begin{pmatrix} n-i \\ n \end{pmatrix}\sum_{j=1}^{2(n+1-i)} g_i(y)\frac{x^{2(n+1-i)-j}}{(2(n+1-i)-j)!}, \tag{46}$$

where $g_i(y)$ is an arbitrary functions of y.

Therefore we have

$$\phi(x,y,t) = \sum_{i=0}^{n}[(-1)^{n-i}(n-i)!\begin{pmatrix} n-i \\ n \end{pmatrix}\sum_{j=1}^{2(n+1-i)} g_i(y)\frac{x^{2(n+1-i)-j}}{(2(n+1-i)-j)!}]t^i. \tag{47}$$

Thus we obtain a rational solutions of (1) and (2) from (30) and (47):

$$H(x,y,t) = \frac{\sum_{i=0}^{n}[(-1)^{n-i}(n-i)!\begin{pmatrix} n-i \\ n \end{pmatrix}\sum_{j=1}^{2(n-i)+1)} g_i(y)\frac{x^{2(n-i)+1-j}}{(2(n-i)+1-j)!}]t^i}{\sum_{i=0}^{n}[(-1)^{n-i}(n-i)!\begin{pmatrix} n-i \\ n \end{pmatrix}\sum_{j=1}^{2(n+1-i)} g_i(y)\frac{x^{2(n+1-i)-j}}{(2(n+1-i)-j)!}]t^i}, \tag{48a}$$

$$G(x,y,t) = \frac{\sum_{i=0}^{n}[(-1)^{n-i}(n-i)!\begin{pmatrix} n-i \\ n \end{pmatrix}\sum_{j=1}^{2(n-i)+1)} g_i(y)\frac{x^{2(n-i)+1-j}}{(2(n-i)+1-j)!}]t^i}{\sum_{i=0}^{n}[(-1)^{n-i}(n-i)!\begin{pmatrix} n-i \\ n \end{pmatrix}\sum_{j=1}^{2(n+1-i)} g_i(y)\frac{x^{2(n+1-i)-j}}{(2(n+1-i)-j)!}]t^i}$$

$$-\frac{\sum_{i=0}^{n}[(-1)^{n-i}(n-i)!\begin{pmatrix} n-i \\ n \end{pmatrix}\sum_{j=1}^{2(n-i)+1)} g_i(y)\frac{x^{2(n-i)+1-j}}{(2(n-i)+1-j)!}]t^i}{\sum_{i=0}^{n}[(-1)^{n-i}(n-i)!\begin{pmatrix} n-i \\ n \end{pmatrix}\sum_{j=1}^{2(n+1-i)} g_i(y)\frac{x^{2(n+1-i)-j}}{(2(n+1-i)-j)!}]t^i}$$

$$\cdot\frac{\sum_{i=0}^{n}[(-1)^{n-i}(n-i)!\begin{pmatrix} n-i \\ n \end{pmatrix}\sum_{j=1}^{2(n-i+1)} g_i'(y)\frac{x^{2(n-i+1-j}}{(2(n-i)+1-j)!}]t^i}{\sum_{i=0}^{n}[(-1)^{n-i}(n-i)!\begin{pmatrix} n-i \\ n \end{pmatrix}\sum_{j=1}^{2(n+1-i)} g_i(y)\frac{x^{2(n+1-i)-j}}{(2(n+1-i)-j)!}]t^i}. \tag{48b}$$

In particular, when $n = 1$, the solutions (48a) and (48b) reduce to

$$H_6(x,y,t) = \frac{g_1(y)(x+t) + g_2(y)}{[g_1(y)x + g_2(y)]t + \frac{1}{2}g_1(y)x^2 + g_2(y)x + g_3(y)]}, \tag{49a}$$

$$G_6(x,y,t) = \frac{g_{1y}(y)(x+t) + g_{2y}(y)}{[g_1(y)x + g_2(y)]t + \frac{1}{2}g_1(y)x^2 + g_2(y)x + g_3(y)]}$$

$$-\frac{[g_1(y)(x+t) + g_2(y)][(g_{1y}(y)x + g_{2y}(y))t + \frac{1}{2}g_{1y}(y)x^2 + g_{2y}(y)x + g_{3y}(y)]}{[(g_1(y)x + g_2(y))t + \frac{1}{2}g_1(y)x^2 + g_2(y)x + g_3(y)]^2}, \tag{49b}$$

where $g_1(y)$ and $g_2(y)$ are functions of y.

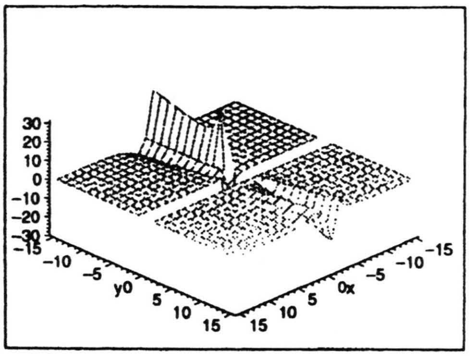

Fig. 9

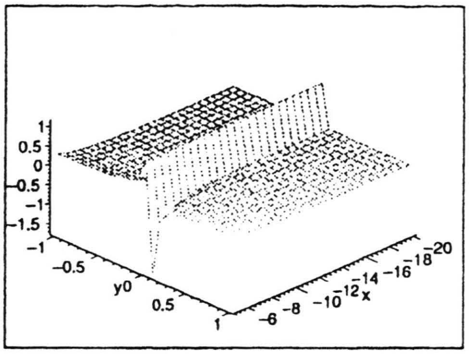

Fig. 10

Figure 9 is a plot of (49a) and shows a singular solution in (2+1)-dimensions at $t = 0$ with $g_1 = y^3$, $g_2 = 3y^2$, $g_3 = 3y$, $h_0 = 0$. Figure 10 is a plot of (49b) and shows a singular soliton solution in (2+1)-dimensions at $t = 0$ with the conditions as Fig.9. These singular solutions develop a singularity at a finite point, i.e., for the fixed $t = 0$, there always exist $x = x_0, y = y_0$ at which the solutions blow up.

4. Conclusions

In summary, we have obtained the multiple soliton-like solutions containing multiple soliton solutions and infinite rational solutions for the Broer-Kaup equation in (2+1)-dimensional space. These solutions may play an important role for the explanation of some practical physical problems. The obtained singular solutions develop a singularity at a finite point, i.e., for any fixed $t = t_0$ there always exists an $x = x_0$ at which the solutions blow up. There is much current interest in the so-called "hot spots" or "blow-up" phenomena [12 - 14]. It appears that the singular solution will model the physical phenomena. Of course, the way can be also extended to other nonlinear wave equations, such as the (2+1)-extension of the classical Boussinesq system [15]. This should be further considered.

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